

# World-sheet dynamics of ZZ branes

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## Abstract

We show how non-compact space-time (ZZ branes) emerges as a limit of compact space-time (FZZT branes) for specific ratios between the square of the boundary cosmological constant and the bulk cosmological constant in the  $(2, 2m-1)$  minimal model coupled to two-dimensional quantum gravity.

# 1 Introduction

Non-critical string theory serves as a good laboratory for the study of non-perturbative effects in string theory. Since non-critical string theory can also be viewed as two-dimensional gravity coupled to matter, it also serves as a model for quantum gravity.

Most recently the dynamics of D-branes was studied in non-critical string theory with  $c < 1$  [1, 2, 3], where  $c$  denotes the central charge of the conformal field theory coupled to quantum gravity. The starting point for this new development was the work of Zamolodchikov and Zamolodchikov (ZZ) [4], who from a purely 2d gravity point of view asked if it is possible to quantize non-compact 2d Euclidean geometries. The consistency conditions imposed on the quantization of the Lobachevskiy-plane (the pseudo-sphere) led to the discovery of the boundary conditions at infinity, which later, in [1, 2, 3], were reinterpreted as branes (the so-called ZZ branes) in the context of non-critical string theory.

There exists an intriguing relation between the ZZ branes of the non-compact pseudo-sphere and the more conventional boundary conditions on compact geometries analyzed by Fateev, Zamolodchikov and Zamolodchikov and by Tachikawa (the FZZT branes) [5], a relation first realized by Martinec [1] and by Seiberg and Shih [2], see also [6]. It is the purpose of this article to provide an interpretation of this relation from the viewpoint of the world-sheet theory, i.e. we ask what is the physics behind the transition from the compact world-sheet geometries characterizing the FZZT branes to the non-compact geometries characterizing the ZZ branes. This world-sheet perspective is interesting from a 2d quantum gravity point of view.

## 2 From compact to non-compact geometry

The general disk and cylinder amplitudes in non-critical string theory were first calculated using matrix model techniques. In order to compare with continuum calculations, performed in the context of Liouville theory, it is necessary to work in the so-called conformal background [7]. In the following we will, for simplicity, concentrate on the disk and the cylinder amplitudes in the  $(2, 2m-1)$  minimal conformal field theories coupled to 2d quantum gravity, also called  $(2, 2m-1)$  non-critical string theories. The disk amplitude, calculated from the one-matrix model, is [8]:

$$w(x) = (-1)^m \hat{P}_m(x, \sqrt{\mu}) \sqrt{x + \sqrt{\mu}} = (-1)^m (\sqrt{\mu})^{(2m-1)/2} P_m(t) \sqrt{t+1}, \quad (1)$$

where  $t = x/\sqrt{\mu}$  and where the polynomial  $P_m(t)$  is of degree  $m-1$ . In the conformal background it is determined by [7]

$$P_m^2(t) (t+1) = 2^{2-2m} (T_{2m-1}(t) + 1) \quad (2)$$

$T_p(t)$  being the first kind of Chebyshev polynomial of degree  $p$ . In eq. (1)  $x$  denotes the boundary cosmological coupling constant and  $\mu$  the bulk cosmological coupling constant, the theory viewed as 2d quantum gravity coupled to the  $(2, 2m-1)$  minimal CFT. The zeros of  $P_m(t)$  are all located on the real axis between  $-1$  and  $1$  and more explicitly we can write:

$$P_m(t) = \prod_{n=1}^{m-1} (t - t_n), \quad t_n = -\cos\left(\frac{2n\pi}{2m-1}\right), \quad 1 \leq n \leq m-1. \quad (3)$$

The zeros of  $P_m(t)$  can be associated with the  $m-1$  principal ZZ branes in the notation of [2].

The so-called loop-loop propagator  $G_\mu(l_1, l_2; d)$  [10, 11, 12, 13] is well suited to show the transition from a compact to a non-compact space. It describes the amplitude of an “exit” loop of length  $l_2$  to be separated a distance  $d$  from an “entrance” loop of length  $l_1$  (the entrance loop conventionally assumed to have one marked point). By Laplace transformation one can introduce the boundary cosmological constants  $x, y$  of the entrance and the exit loops, i.e. FZZT-branes on both boundaries:

$$G_\mu(x, y; d) = \int_0^\infty \int_0^\infty dl_1 dl_2 e^{-l_1 x} e^{-l_2 y} G_\mu(l_1, l_2; d). \quad (4)$$

It can be shown [10, 11, 12] that  $G_\mu(x, y; d)$  satisfies the following equation:

$$\frac{\partial}{\partial d} G_\mu(x, y; d) = -\frac{\partial}{\partial x} w(x) G_\mu(x, y; d), \quad (5)$$

with the following solution:

$$G_\mu(x, y; d) = \frac{w(\bar{x}(d))}{w(x)} \frac{1}{\bar{x}(d) + y}, \quad (6)$$

where the so-called running boundary coupling constant  $\bar{x}(d)$  is the solution of the characteristic equation corresponding to (5), i.e.

$$d = \int_{\bar{x}(d)}^x \frac{dx'}{w(x')} \quad (7)$$

While  $G_\mu(x, y; d)$  is not a much studied object in 2d quantum gravity, it is actually a kind of fundamental building block: In pure quantum gravity knowing  $G_\mu(x, y; d)$  allows one to calculate the cylinder amplitude  $C(x, y; \mu)$  and in principle all higher loop functions  $C(x_1, \dots, x_n; \mu)$  [14].

In the case of pure 2d quantum gravity, i.e.  $c=0$  in the terminology of non-critical string theory,  $d$  measures the geodesic distance on the underlying geometries in the path integral and  $G_\mu(x, y; d)$  can be given the following interpretation for  $x \rightarrow \infty$ : It is the amplitude for a disk where the boundary (with boundary cosmological constant  $y$ ) is located a geodesic distance  $d$  from the “center” (the other boundary contracted to a point). This amplitude is difficult to address in Liouville theory because it is difficult to work with the geodesic distance, which in the Liouville setup is a derived non-local concept. However, the combinatorial approach pioneered in [10] allows a transparent derivation of eq. (5). As shown in [9] the disk amplitude arising from  $G_\mu$  offers insight into the transition from compact to non-compact worldsheet geometry. From (7) one observes that when  $d \rightarrow \infty$  the running boundary coupling constant  $\bar{x}(d)$  converges to the zero  $x_0$  of  $\hat{P}_2(x)$ , which in notation of [2] is related to the single principal ZZ-brane in pure quantum gravity. Moreover, for  $y = -x_0$  one obtains the “quantum” Poincare disk when  $d \rightarrow \infty$ . Hence, for this particular value of  $y$  one has a transition from a FZZT brane to a ZZ brane. Notice, this transition is not generic: The average area and the average boundary length of the disk remain finite in the limit  $d \rightarrow \infty$  for all other values of  $y$ .

For the  $(2, 2m-1)$  minimal model coupled to 2d gravity (6) reads:

$$G_\mu(t, t'; d) \propto \frac{1}{\sqrt{\mu}} \frac{1}{\bar{t}(d) + t'} \frac{\sqrt{1 + \bar{t}(d)} \prod_{n=1}^{m-1} (\bar{t}(d) - t_n)}{\sqrt{1 + t} \prod_{n=1}^{m-1} (t - t_n)} \quad (8)$$

where we use the notation of (1), i.e.  $t = x/\sqrt{\mu}$ ,  $t' = y/\sqrt{\mu}$  and  $\bar{t}(d) = \bar{x}(d)/\sqrt{\mu}$ , where  $\bar{x}(d)$  is defined by eq. (7).  $d$  is not the geodesic distance for  $m > 2$ . Rather, it is a distance measured in terms of matter excitations. This is explicit by construction in some models of quantum gravity with matter, for instance the Ising model and the  $c=-2$  model formulated as an  $O(-2)$  model [16, 15]. However, we can still use  $d$  as a measure of distance and we will do so in the following. When  $d \rightarrow \infty$  it follows from (7) that the running boundary coupling constant  $\bar{t}(d)$  converges to one of the zeros of the polynomial  $P_m(t)$ , i.e.

$$\bar{t}(d) \xrightarrow{d \rightarrow \infty} t_k, \quad t_k = -\cos\left(\frac{2k\pi}{2m-1}\right). \quad (9)$$

The cylinderamplitude (8) vanishes for generic values of  $t'$  in the limit  $d \rightarrow \infty$ . However, as shown in [9] we have a unique situation when we choose  $t' = -t_k$  since in this case the term  $1/(\bar{t}(d) + t')$  in (8) becomes singular for  $d \rightarrow \infty$ . After

some algebra we obtain the following expression:

$$G_\mu(t, t' = -t_k, d \rightarrow \infty) \propto \frac{1}{\sqrt{\mu}} \frac{1}{\sqrt{1+t}} \sum_{n=1}^{m-1} (-1)^n \sin\left(\frac{2n\pi}{2m-1}\right) \left[ \frac{1}{\sqrt{1+t} + \sqrt{1+t_n}} - \frac{1}{\sqrt{1+t} - \sqrt{1+t_n}} \right]. \quad (10)$$

Note,  $G_\mu(t, t' = -t_k, d \rightarrow \infty)$  is independent of which zero  $t_k$  the running boundary coupling constant approaches in the limit  $d \rightarrow \infty$ , apart from an overall constant of proportionality.

Formula (10) describes an AdS-like non-compact space with cosmological constant  $\mu$  and with one compact boundary with boundary cosmological constant  $x$  as explained in [9] in the case of pure gravity. In the last section we will comment on the fact that we have to set  $t' = -t_k$  in order to generate an AdS-like non-compact space in the limit  $d \rightarrow \infty$ . Now, we will explain how the cylinder amplitude (10) is related to the conventional FZZT–ZZ cylinder amplitude in the Liouville approach to quantum gravity.

### 3 The cylinder amplitudes

The  $(2, 2m-1)$  minimal CFT coupled to 2d quantum gravity has a one-matrix representation. Using the one-matrix model one can calculate the disk amplitude like (1) or the cylinder amplitude. Quite remarkable, the cylinder amplitude is “universal”, i.e. the same in all the  $(2, 2m-1)$  minimal models coupled to quantum gravity [17, 7]:

$$C_\mu(t_1, t_2) = \frac{1}{2\mu} \frac{1}{(\sqrt{t_1+1} + \sqrt{t_2+1})^2} \frac{1}{\sqrt{(t_1+1)(t_2+1)}}, \quad (11)$$

where  $t_1, t_2$  are boundary cosmological constants (divided by  $\sqrt{\mu}$ ) and  $\mu$  is the bulk cosmological constant. This amplitude is one where both boundaries are marked, i.e. differentiated after the boundary cosmological constants, and the corresponding unmarked amplitude  $Z$  is:

$$C_\mu(t_1, t_2) = \frac{1}{\mu} \frac{\partial^2}{\partial t_1 \partial t_2} Z_\mu(t_1, t_2), \quad (12)$$

$$Z_\mu(t_1, t_2) = -\log \left[ \left( \sqrt{t_1+1} + \sqrt{t_2+1} \right)^2 \sqrt{\mu} a \right], \quad (13)$$

where  $a$  is a (lattice) cut-off.

The amplitude  $Z_\mu(t_1, t_2)$  is only one of many cylinder amplitudes which in principle exist when we consider a  $(2, 2m-1)$  minimal conformal field theory

coupled to 2d gravity. If we consider the cylinder amplitude of the  $(2, 2m-1)$  minimal conformal field theory before coupling to gravity we have available  $m-1$  Cardy boundary states  $|r\rangle_{\text{Cardy}}$ ,  $r=1, \dots, m-1$ , on each of the boundaries, and a corresponding cylinder amplitude for each pair of Cardy boundary states [18]:

$$Z_{\text{matter}}(r, s; q) = \sqrt{2} b \sum_{l=1}^{m-1} (-1)^{r+s+m+l+1} \frac{\sin(\pi r l b^2) \sin(\pi s l b^2)}{\sin(\pi l b^2)} \chi_l(q), \quad (14)$$

where

$$b = \sqrt{\frac{2}{2m-1}} \quad (15)$$

and where we consider a cylinder with a circumference of  $2\pi$  and length  $\pi\tau$  in the closed string channel. The generic non-degenerate Virasoro character  $\chi_p(q)$  is

$$\chi_p(q) = \frac{q^{p^2}}{\eta(q)}, \quad q = e^{-2\pi\tau}, \quad (16)$$

where  $\eta(q)$  is the Dedekind function. However, the degenerate Virasoro character  $\chi_l(q)$  in eq. (14) is given by [19]:

$$\chi_l(q) = \frac{1}{\eta(q)} \sum_{n \in \mathbb{Z}} \left( q^{(2n/b+1/2(1/b-lb))^2} - q^{(2n/b+1/2(1/b+lb))^2} \right). \quad (17)$$

In order to couple the cylinder amplitude in eq. (14) to 2d quantum gravity one has, in the conformal gauge, to multiply  $Z_{\text{mat}}(r, s; q)$  by a contribution  $Z_{\text{ghost}}(q)$  obtained by integrating over the ghost field, as well as by a contribution  $Z_{\text{Liouv}}(t_1, t_2; q)$  obtained by integrating over the Liouville field. Explicitly we have

$$Z_{\text{ghost}}(q) = \eta^2(q), \quad Z_{\text{Liouv}}(t_1, t_2; q) = \int_0^\infty dP \bar{\Psi}_{\sigma_1}(P) \Psi_{\sigma_2}(P) \chi_P(q), \quad (18)$$

where  $\Psi_\sigma(P)$  is the FZZT boundary wave function [5], such that

$$\bar{\Psi}_{\sigma_1}(P) \Psi_{\sigma_2}(P) = \frac{4\pi^2 \cos(2\pi P \sigma_1) \cos(2\pi P \sigma_2)}{\sinh(2\pi P/b) \sinh(2\pi P b)}, \quad (19)$$

and where  $\sigma$  is related to the boundary cosmological constant by

$$\frac{x}{\sqrt{\mu}} \equiv t = \cosh(\pi b \sigma). \quad (20)$$

One finally obtains the full cylinder amplitude by integrating over the single real moduli  $\tau$  of the cylinder:

$$Z_\mu(r, t_1; s, t_2) = \int_0^\infty d\tau Z_{\text{ghost}}(q) Z_{\text{Liouv}}(t_1, t_2; q) Z_{\text{mat}}(r, s; q). \quad (21)$$

This cylinder amplitude depends not only on the Cardy states  $r, s$ , but also on the values of the boundary cosmological constants  $t_1, t_2$  as well as the bulk cosmological constant  $\mu$ .

From the discussion above it is natural that the matrix model (for a specific value of  $m$ ) only leads to a single cylinder amplitude since it corresponds to an explicit (lattice) realization of the conformal field theory, and thus only to one realization of boundary conditions. In the language of Cardy states we want first to identify *which* boundary condition is realized in the scaling limits of the one-matrix model. We do that by calculating the cylinder amplitude (21) and then comparing the result with the matrix model amplitude.

The calculation, using (14), (18) and (21), is in principle straight forward, but quite tedious. The main technical problem is to regularize the  $P$ -integration around zero<sup>1</sup> and to perform suitable deformations at infinity. This can be done following [3], and the details will be reported elsewhere [20]. The result is for  $r + s \leq m$

$$Z_\mu(r, t_1; s, t_2) = - \sum_{k=1-r}^{r-1}{}' \sum_{l=1-s}^{s-1}{}' \log \left( \left[ (\sqrt{t_1+1} + \sqrt{t_2+1})^2 - f_{k,l}(t_1, t_2) \right] \sqrt{\mu} a \right) \quad (22)$$

where  $a$  is the cut-off (as in (12)) and the summations are in steps of two, indicated by the primes in the summation symbols.

$$f_{k,l}(t_1, t_2) = 4 \left[ \sqrt{(t_1+1)(t_2+1)} + 2 \cos^2 \left( \frac{(k+l)\pi b^2}{4} \right) \right] \sin^2 \left( \frac{(k+l)\pi b^2}{4} \right). \quad (23)$$

From eqs. (22) and (23) it follows that we have agreement with the matrix model amplitude (12) if and only if  $r = s = 1$ . The  $r = 1$  boundary condition is in the concrete realizations of conformal field theories related to the so-called fixed boundary conditions and for the matter part of the cylinder amplitude it corresponds to the fact, that only the conformal family of states associated with the identity operator propagates in the open string channel.

It is worth to notice that introducing the variables  $\sigma_1, \sigma_2$  from eq. (20) instead of  $t_1, t_2$  in eq. (22) the cylinder amplitudes can be rewritten as:

$$Z_\mu(r, \sigma_1; s, \sigma_2) = \sum_{k=1-r}^{r-1}{}' \sum_{l=1-s}^{s-1}{}' Z_\mu(1, \sigma_1 + ibk; 1, \sigma_2 + ibl), \quad (24)$$

which shows that for  $r + s \leq m$  one can express the cylinder amplitudes  $Z_\mu(r; s)$  as superpositions of the  $Z_\mu(1; 1)$  amplitudes with *complex* values of the boundary

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<sup>1</sup>One can avoid this by working with  $C(x_1, x_2)$ . However, we prefer to work directly with  $Z(x_1, x_2)$  in order to compare our results with previous calculations.

cosmological constants. The composition (24) of the cylinder amplitude is fully consistent with a similar decomposition of the disk amplitude made in [2]. However, (24) is not valid for  $m < r + s \leq 2(m-1)$ . Assuming the validity of (24) for all values of  $r + s \leq 2(m-1)$  and of (22) for  $r = s = 1$ , which coincides with the matrix model amplitude, is not consistent with the vanishing of the cylinder amplitude with two different matter Ishibashi states imposed on the boundaries. From the fusion rules one can derive the correct expression for  $Z_\mu(r, \sigma_1; s, \sigma_2)$  for  $r + s > m$  expressed in terms of  $Z_\mu(r, \sigma_1; s, \sigma_2)$  with  $r + s \leq m$ , an expression which can also be obtained by direct calculation. The explicit expression will be reported elsewhere [20]. Here, we only note that any conclusion we report in this article is valid also for  $r + s > m$  unless otherwise stated.

Following Martinec [1] it is now possible to calculate the FZZT–ZZ amplitude by replacing one of the FZZT wave functions in (18) with

$$\Psi_{\hat{n}}(P) \propto \Psi_{\sigma(\hat{n})}(P) - \Psi_{\sigma(-\hat{n})}(P), \quad (25)$$

where (for the  $(2, 2m-1)$  models)

$$\sigma(\hat{n}) = i \left( \frac{1}{b} + \hat{n} b \right), \quad (26)$$

and where  $\hat{n} = 1, \dots, m-1$  is an integer labeling the different principal ZZ-branes. Notice, the boundary cosmological constants  $t_{\hat{n}}$  and  $t_{-\hat{n}}$  corresponding to the complex valued  $\sigma(\hat{n})$  and  $\sigma(-\hat{n})$  are still real and are actually the same for a given value of  $\hat{n}$ :

$$t_{\hat{n}} = t_{-\hat{n}} = -\cos \left( \frac{2\hat{n}\pi}{2m+1} \right), \quad (27)$$

i.e. they are the zeros of the polynomial  $P_m(t)$  in formula (1). We now obtain the following FZZT–ZZ cylinder amplitude<sup>2</sup> for  $r + s \leq m$ , differentiated after the boundary cosmological constant on the FZZT brane:

$$Z'_\mu(r, \hat{n}; s, t) \propto \sum_{k=-(r-1)}^{r-1} \sum_{l=-(s-1)}^{s-1} \frac{(\pm)}{\sqrt{\mu}\sqrt{1+t}} \left[ \frac{1}{\sqrt{t+1} + \sqrt{1+t_{k+l+\hat{n}}}} - \frac{1}{\sqrt{t+1} - \sqrt{1+t_{k+l+\hat{n}}}} \right]. \quad (28)$$

The differentiation after the boundary cosmological constant is performed in order to compare with the corresponding amplitude  $G_\mu(t, t' = -t_{\hat{n}}, d \rightarrow \infty)$  given by

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<sup>2</sup>The upper sign in (28) is for  $0 \leq k + l + \hat{n}$ , while the lower sign is for  $k + l + \hat{n} \leq 0$



(10), which is the amplitude of a cylinder with one marked point on the compact boundary.

Let us now consider the FZZT-ZZ cylinder amplitude with an  $r = 1$  Cardy matter boundary condition imposed on the FZZT boundary. This is the natural choice if we want to compare with the matrix model results since the Cardy matter boundary condition captured by the matrix model is precisely  $r = 1$ . In this case the summation over  $s$  is not present in eq. (28) and comparing formula (28) with the expression (10) for  $G_\mu(t, -t_{\hat{n}}, d \rightarrow \infty)$  one can show that

$$G_\mu(t, -t_{\hat{n}}, d \rightarrow \infty) \propto \sum_{r=1}^{m-1} S_{1,r} Z'_\mu(r, \hat{n}; 1, t), \quad (29)$$

where  $S_{k,l}$  is the modular S-matrix in the  $(2, 2m-1)$  minimal CFT, i.e. [19]

$$S_{k,l} = \sqrt{2} b (-1)^{m+k+l} \sin(\pi k l b^2). \quad (30)$$

This result is valid for any  $(2, 2m-1)$  minimal CFT coupled to quantum gravity and is valid independent of which zero  $t_k$  the running boundary coupling constant approaches in the limit  $d \rightarrow \infty$ . The proof of (29) is straight forward but tedious and will appear elsewhere [20].

The natural interpretation of eq. (29) is that the matter boundary state of the exit loop in the loop-loop amplitude  $G_\mu(t, -t_{\hat{n}}, d)$  is projected on the following linear combination of Cardy boundary states in the limit  $d \rightarrow \infty$ :

$$|a\rangle = \sum_{r=1}^{m-1} S_{1,r} |r\rangle_{\text{Cardy}} \propto |1\rangle, \quad (31)$$

where the last state is the Ishibashi state corresponding to the identity operator and where we have used the orthogonality properties of the modular S-matrix and the relation between Cardy states and Ishibashi states:

$$|r\rangle_{\text{Cardy}} = \sum_{k=1}^{m-1} \frac{S_{r,k}}{\sqrt{S_{1,k}}} |k\rangle. \quad (32)$$

The Ishibashi state corresponding to the identity operator is in a certain way the simplest boundary state available, and it is remarkable that it is precisely this state which is captured by the explicit transition from compact to non-compact geometry enforced by taking the distance  $d \rightarrow \infty$ .

## 4 Discussion

We have shown how it is possible to construct an explicit transition from compact to non-compact geometry in the framework of 2d quantum gravity coupled to

conformal field theories. The non-compact geometry is AdS-like in the sense that the average area and the average length of the exit loop diverge exponentially with  $d$  when  $d \rightarrow \infty$  as shown in [9] (for pure gravity), and the corresponding amplitude can be related to the FZZT-ZZ cylinder amplitude with the simplest Ishibashi state living on the ZZ brane. The  $d \rightarrow \infty$  limit plays an instrumental role and we would like to address two important aspects of this.

Firstly, in [2] it was advocated that the algebraic surface

$$T_p(w/C_{p,q}(\mu)) = T_q(t), \quad (33)$$

where  $C_{p,q}(\mu)$  is a constant, is the natural "target space" of  $(p, q)$  non-critical string theory. For  $(p, q) = (2, 2m-1)$  eq. (33) reads

$$w^2 = \mu^{\frac{2m-1}{2}} P_m^2(t)(t+1), \quad (34)$$

and in this case the extended target space is a double sheeted cover of the complex  $t$ -plane except at the singular points, which are precisely the points  $(t_k, w=0)$  associated with the zeros of the polynomial  $P_m(t)$ . One is also led to this extended target space from the world-sheet considerations made here. We want the running boundary coupling constant to be able to approach any of the fixed points  $t_k$  in the limit  $d \rightarrow \infty$ , i.e. we want all the fixed points to be attractive. This is only possible if we consider the running boundary coupling constant  $\bar{t}(d) = \bar{x}(d)/\sqrt{\mu}$  defined in eq. (7) as a function taking values on the algebraic surface defined by (34). The reason is that  $t_k$  is either an attractive or a repulsive fixed point depending on which sheet we consider and some of the fixed points are attractive on one sheet, while the other fixed points are attractive on the other sheet. Hence, we are forced to view  $\bar{t}(d)$  as a map to the double sheeted Riemann surface defined by eq. (34) in the  $(2, 2m-1)$  minimal model coupled to quantum gravity.

The picture becomes particularly transparent if we use the uniformization variable  $z$  introduced for the  $(p, q)$  non-critical string in [2] by

$$t = T_p(z), \quad w/C_{p,q}(\mu) = T_q(z), \quad (35)$$

i.e. in the case of  $(p, q) = (2, 2m-1)$ :

$$z = \frac{1}{\sqrt{2}} \sqrt{t+1}. \quad (36)$$

The map (35) is one-to-one from the complex plane to the algebraic surface (33), except at the singular points of the surface where it is two-to-one. The singular points are precisely the points corresponding to ZZ branes. If we change variables from  $x$  to  $z$  in eq. (5) (choosing  $\mu=1$  for simplicity) we obtain

$$\frac{\partial}{\partial d} \tilde{G}_\mu(z, z'; d) = -\frac{\partial}{\partial z} \tilde{P}_m(z) \tilde{G}_\mu(z, z'; d), \quad (37)$$

where  $\tilde{G}_\mu(z, z'; d) = zG_\mu(x, y; d)$  and where the polynomial  $\tilde{P}_m(z)$  is

$$\tilde{P}_m(z) \propto \prod_{k=1}^{m-1} (z^2 - z_k^2), \quad z_k = \sin\left(\frac{\pi}{2} b^2 k\right). \quad (38)$$

Each zero  $t_k$  of  $P_m(t)$  gives rise to two zeros  $\pm z_k$  of  $\tilde{P}_m(z)$ . The zeros  $\pm z_k$  are the fixed points of the running “uniformized” boundary cosmological constant  $\bar{z}$  associated with the characteristic equation corresponding to eq. (37). For a given value of  $k$  one of the two zeros  $\pm z_k$  is an attractive fixed point, while the other is repulsive. Moving from one sheet to the other sheet on the algebraic surface (34) corresponds to crossing the imaginary axis in the  $z$ -plane. Hence, for a given value of  $k$  the two fixed points  $\pm z_k$  are each associated with a separate sheet and  $\bar{z}$  will only approach the attractive of the two fixed points  $\pm z_k$ , if  $\bar{t}(d)$  belongs to the correct sheet.

Quite remarkable eq. (37) was derived in the case of pure 2d gravity (the (2, 3) model corresponding to  $c = 0$ ) using a completely different approach to quantum gravity called CDT<sup>3</sup> (causal dynamical triangulations) [21] and the uniformization transformation relating the CDT boundary cosmological constant  $z$  to the boundary cosmological constant  $t$  was derived and given a world-sheet interpretation in [22], but again from a different perspective. From the CDT loop-loop amplitude determined by (37) one can define a CDT “ZZ brane” with non-compact geometry [25].

Secondly, our construction also adds to the understanding of the relation (25) discovered by Martinec. In Liouville theory there is a one-to-one correspondance between the ZZ boundary states labeled by  $(m, n)$  and the degenerate primary operators  $V_{m,n}$  [4]. This correspondance completely determines the Liouville cylinder amplitude with two ZZ boundary conditions: The spectrum of states flowing in the open string channel between two ZZ boundary states is obtained from the fusion algebra of the corresponding degenerate operators. Similarly, there is a one-to-one correspondance between the FZZT boundary states labeled by  $\sigma > 0$  and the non-local “normalizable” primary operators  $V_\sigma = \exp((Q + i\sigma)\phi)$ , where  $\phi$  is the Liouville field. The conformal dimension of the spin-less degenerate primary operator  $V_{m,n}$  is given by

$$\Delta_{m,n} = \frac{Q^2 - (m/b + nb)^2}{4}, \quad (39)$$

while the conformal dimension of the spin-less non-local primary operator  $V_\sigma$  is given by

$$\Delta_\sigma = \frac{Q^2 + \sigma^2}{4}. \quad (40)$$

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<sup>3</sup>It should be noted that CDT seemingly has an interesting generalization to higher dimensional quantum gravity theories [23, 24]

Since  $\Delta_{m,n} = \Delta_\sigma$  for  $\sigma = i(m/b + nb)$ , one is naively led to the wrong conclusion, that a FZZT boundary state turns into a ZZ boundary state, if one tunes  $\sigma = i(m/b + nb)$ . However, the operator  $V_{m,n}$  is degenerate and in addition to setting  $\sigma = i(m/b + nb)$  we therefore have to truncate the spectrum of open string states, that couple to the FZZT boundary state, in order to obtain a ZZ boundary state. This is precisely captured in the relation (25) concerning the principal ZZ boundary states. The world-sheet geometry characterizing the FZZT brane is compact, while the world-sheet geometry of the ZZ-brane is non-compact. Hence, truncating the spectrum of open string states induces a transition from compact to non-compact geometry. In our concrete realization of this transition this truncation is obtained by first setting the boundary cosmological constant  $t' = -t_{\hat{n}}$  on the exit loop and then taking  $d \rightarrow \infty$ . It is interesting to note that in the original articles introducing the FZZT and ZZ boundary states [5, 4] the square of eq. (20) is always used as the defining relation between  $\sigma$  and  $t$ . Thus, in these works both  $\pm t_{\hat{n}}$  are associated with the ZZ brane labeled by  $(1, \hat{n})$  through eq. (25). In our explicit construction both values also play a role:  $t_{\hat{n}}$  is the fixed point of the running boundary coupling constant  $\bar{t}(d)$  as  $d \rightarrow \infty$  and  $-t_{\hat{n}}$  is the actual value (measured in units of  $\sqrt{\mu}$ ) of the boundary cosmological constant on the "AdS-boundary". Only for this particular value of the boundary cosmological constant does an AdS geometry emerge in the limit  $d \rightarrow \infty$ .

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